EVALUATING THE CONDUCTIVITY OF PLANE FIGURES UNDER BOUNDARY CONDITIONS OF THE THIRD KIND

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The separation method to evaluate the conductivity of plane figures is extended to the case of boundary conditions of the third kind. Examples are presented for the application of this method.

A method is described in [1] for the evaluation of conductivity in plane figures with isothermal boundaries; this method is based on the separation of these figures by means of isothermal and adiabatic lines. Later on we will examine the more general case in which boundary conditions of the third kind are specified for the boundaries of the figure.

First of all, let us reduce the problem to some thermal-conductivity case for a figure with isothermal boundaries. It is most natural to replace the thermal heat-transfer resistance by the resistance of a thin film adhering to the boundary. The outside boundary of the film should be regarded as isothermal, the resistance across the film should be specified and finite, and the resistance along the film must be infinitely large. This is the generally accepted model. Let us introduce one more equivalent transformation of the model to achieve a uniform figure. We will replace the film with a "rake" of the basic material of the figure so that the film element – of infinitesimally small length dl – changes into a rectangle of width dl and height λ/α (see Fig. 1a). The external base of the rectangle will be treated as the isothermal boundary, and the sides will be regarded as adiabatic boundaries. Because of this last condition, we satisfy the requirement for infinitely great thermal resistance along the "rake." The adjacent rectangles may abut each other, but this should have no effect on their overall conductivity, i.e., the coincident segments of the surfaces on adjacent rectangles must be treated as if they were two-layered (see Fig. 1b).

The basic theorem with regard to the separation of plane figures [1] and the corresponding method of evaluating conductivity are applicable to the figure derived in this manner. We find the upper bound by drawing the isothermal line of separation along the inside boundary of the "rake," i.e., along the outside boundary of the original figure. Then, by the definition of conductivity and according to certain familiar rules of addition for series and parallel-connected resistances, we have

$$\Pi_\alpha = \frac{Q}{\lambda \Delta t} < \left[\frac{\lambda}{\alpha_1 l_1} + \frac{1}{\Pi} + \frac{\lambda}{\alpha_2 l_2} \right]^{-1}. \tag{1}$$

Here Q is the heat flow per unit length of the body whose lateral cross section is a specified plane figure. Formula (1) is well known; however, it is generally not stressed in the literature that it always yields an exaggerated result.

For the lower bound of the conductivity Π_α we have to separate the equivalent figure into strips by means of adiabatic lines, extending these to the boundaries of the original figure, and then we have to sum the conductivities of these strips. Near the boundaries of the original figure the strips change into the teeth of a "rake," of which we spoke earlier. As an example, let us find the bound Π_α for a rectangular trapezoid for which the heat transfer is accomplished through the base, with the sides serving as the adiabatic boundaries. The lines of separation are drawn precisely as in [1]. For the elementary strip inequality (1) changes into an equality, and here we should note that l_1, Π, and l_2 must be replaced, respectively, by $dl_1 = da$, $d\Pi$, and $dl_2 = db$. Using this equality and certain geometric relationships, we find

$$\Pi_\alpha > \int_0^\beta \frac{d\varphi}{\dfrac{\lambda \cos^2\varphi}{\alpha_a a \operatorname{ctg}\beta} + \ln\dfrac{b}{a} + \dfrac{\lambda\cos^2\varphi}{\alpha_b b \operatorname{ctg}\beta}}.$$

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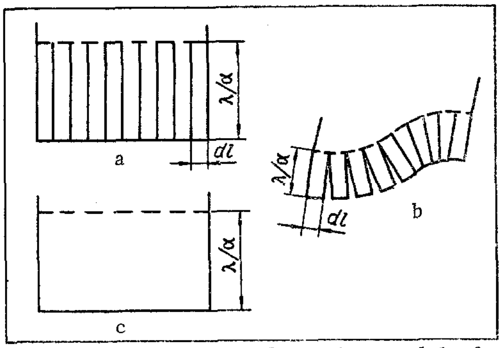

Fig. 1. Thermal-conductivity models for the heat transfer at the boundaries of the figure (the boundary of the original figure is shown by the dashed line): a) in the form of a "rake" on a rectilinear boundary; b) in the form of a "rake" at a curvilinear boundary; c) a model of an additional layer.

The integration variable φ denotes the angle formed by the straight line of separation with the smaller side of the trapezoid. Introducing the notations $m = \ln(b/a)$ and $n = \lambda \tan\beta(1/\alpha_a a + 1/\alpha_b b)$ and integrating, we find the following bound for the conductivity of a rectangular trapezoid:

$$\Pi_\alpha > \frac{1}{\sqrt{m(m+n)}} \operatorname{arctg} \frac{\sqrt{m}\,\mathrm{tg}\beta}{\sqrt{m+n}}. \tag{2}$$

As $\alpha_a \to \infty$ and $\alpha_b \to \infty$ $(n = 0)$ it follows from formula (2) that

$$\Pi_\alpha > \frac{\beta}{\ln\frac{b}{a}},$$

i.e., the conventional bound for a rectangular trapezoid with isothermal bases [1]. When $a = b$ $(m = 0)$ formula (2) yields

$$\Pi_\alpha = \frac{\mathrm{tg}\,\beta}{n} = a \Big/ \lambda\left(\frac{1}{\alpha_a} + \frac{1}{\alpha_b}\right).$$

In the last expression, we have replaced the > with an equality sign, since formula (1) yields precisely the same value of Π_α and it is therefore exact.

Let us consider another example which pertains to the case of a "tube in a semi-infinite block." The corresponding plane figure has only one of its boundaries in the form of a circle of radius r, while the other boundary is in the form of a straight line whose distance from the center of the circle is denoted h. If the boundaries of the figure are isothermal, the problem of determining the conductivities and the temperature fields admit of an exact analytical solution, e.g., by means of a conformal mapping of this figure in a concentric circle by means of the function $w = 1/z$. The formula for the conductivity [2] has the form

$$\Pi = \frac{2\pi}{\ln\xi}\left(\xi = \eta + \sqrt{\eta^2 - 1};\ \eta = \frac{h}{r}\right). \tag{3}$$

For boundary conditions of the third kind no exact solution is yet possible. The lower bound for the conductivity of the figure under these conditions can be found by separating an equivalent figure by means of adiabatic lines which, within the confines of the original figure, coincide with its adiabatic curves, plotted under the condition that the boundaries are isothermal. In this manner we derived the formula

$$\Pi_\alpha > 2\pi\left\{\left[\ln\xi + \frac{\lambda(\xi+1)}{\alpha_r r(\xi-1)} + \frac{2\lambda}{\alpha_l r \sqrt{\eta^2-1}}\right]\left[\ln\xi + \frac{\lambda(\xi-1)}{\alpha_r r(\xi+1)}\right]\right\}^{-\frac{1}{2}}. \tag{4}$$

We can see immediately that as $\alpha_r \to \infty$ and $\alpha_l \to \infty$ (the isothermal boundaries) expression (4) changes into Eq. (3). When $\eta = \xi = 1$ (the circle in contact with the straight-line boundary) expression (4) becomes

$$\Pi_\alpha > 2\pi\left[\frac{\lambda}{r}\left(2 + \frac{\lambda}{\alpha_r r}\right)\left(\frac{1}{\alpha_r} + \frac{1}{\alpha_l}\right)\right]^{-\frac{1}{2}}.$$

Let us work on the upper bound of Π_α for the figure under consideration. It would be wrong to use formula (1) at this point, since one of the terms in the denominator – relating to the rectilinear boundary $(\lambda/\alpha_l l)$ – vanishes. The resistance of the "rake" adjoining the rectilinear boundary is automatically eliminated in this case and the bound is found to be somewhat too approximate. In this and similar cases, i.e., with an infinite rectilinear boundary, for the approximate calculation of Π_α the literature [2] recommends the method of the "additional layer." However, in the literature sources known to us there is no indication as to the bound for Π_α that results from this method, i.e., whether the upper or the lower bound. To respond to this question, let us compare the accepted model for the boundary conditions of the third kind (see Fig. 1a) with the model of the additional layer (see Fig. 1c). According to the fundamental position of the separation method [1], transitions from model (c) to model (a) must lead to a reduction in conductivity. Consequently, the additional-layer method at a rectilinear boundary yields an upper bound for Π_α.

Using separation by means of an isothermal line on a circular boundary for the figure being investigated, in combination with the additional layer at the rectilinear boundary, in conjunction with (1) and (3) we find

$$\Pi_\alpha < \frac{2\pi}{\ln \xi^* + \frac{\lambda}{\alpha_r r}} \left(\xi^* = \frac{h^*}{r} + \sqrt{\left(\frac{h^*}{r}\right)^2 - 1}\ ;\ \ h^* = h + \frac{\lambda}{\alpha_l} \right).$$

The right-hand member of this expression coincides with the relationship given in the handbook [2].

The conclusion to the effect that the additional layer affects Π_α is also valid for a nonrectilinear convex boundary. Here we should use a method of separation with an isothermal line, or the method of the additional layer, depending on which bound will be better, i.e., which bound will yield a smaller value for Π_α. With a concave boundary, the nature of the bound given by the method of the additional layer is indeterminate, and the use of this method is not recommended. We are dealing here with the fact that in changing from the "rake" to the additional layer, on the one hand, the conductivity is increased because there are no adiabatic boundaries between the teeth, while on the other hand, the conductivity is reduced because of the partial contact between the adjacent teeth.

NOTATION

l is the length of the figure boundary at which the heat transfer takes place;
λ is the thermal conductivity;
α is the heat-transfer coefficient (the subscripts with the α correspond to the designation of the boundary for which the heat-transfer coefficient is specified);
Π_α is the conductivity of the plane figure under boundary conditions of the third kind;
Π is the conductivity of the plane figure with isothermal boundaries;
Q is the heat flow;
Δt is the temperature difference for the media flushing the boundaries of the figure;
a, b are, respectively, the base lengths for the rectangular trapezoid (b > a);
β is the angle formed by the sides of the rectangular trapezoid;
φ is a variable angle;
r is the radius of the circular boundary;
h is the distance from the rectilinear boundary to the center of the circular boundary;
w, z are complex variables;
η, ξ are dimensionless parameters.

LITERATURE CITED

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2. S. S. Kutateladze and V. M. Borishanskii, Heat-Transfer Handbook [in Russian], GÉI, Moscow–Leningrad (1959).